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PLANE BROWNIAN MOTION HAS STRICTLY n-MULTIPLE POINTS

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ABSTRACT

It is shown that, almost surely, for all natural n, there are points which the plane Brownian motion visits *exactly* n times.

A point x is k-multiple (respectively strictly k-multiple) for a map f if, and only if,

$$\operatorname{Card} f^{-1}\{x\} \geqq k \qquad (\text{resp.} = k).$$

Let Z be a plane Brownian motion. (All Brownian motions discussed here are continuous, and defined on $\mathbf{R}_+ \equiv [0,\infty[$.)

Dvoretzky, Erdös and Kakutani proved in [2] that, with probability 1, for all natural n, Z admits n-multiple points. With this at hand, we provide a simple proof to the following

THEOREM. *Almost surely, for all natural n, Z admits strictly n-multiple points.*†

Fix a natural $n \geqq 2$. We shall see that, almost surely, Z admits at least one strictly n-multiple point.

Suppose $Z(\omega)$ has n-multiple points (which, by [2], is the case for almost all ω). So there are n mutually disjoint closed rational subintervals of $\mathbf{R}_+$ whose images under $Z(\omega)$ have a common point of intersection. (A rational interval is one with rational endpoints.) Since the set of finite sets of rational intervals is only countable, it is enough to show that if $I_1, \ldots, I_n$ are mutually disjoint

† One of us proved recently ([1]) that if S is a closed subset of $\mathbf{R}_+$ without interior points, then, almost surely, there exist points in the plane whose inverse image under Z is order-similar to S. This, of course, is stronger than our theorem (to obtain the almost sure existence of strictly n-multiple points, take $S = \{1, \ldots, n\}$), but its proof (the one in [1], at least) is relatively involved.

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compact subintervals of $\mathbf{R}_+$, then, almost surely, if $A \equiv Z(I_1) \cap \cdots \cap Z(I_n)$ is nonempty, then there is a point $p \in A$ which is strictly n-multiple for Z.

Suppose I_k are as above, and set $I = \cup I_k$.

If K is a nonempty compact subset of the plane $\mathbf{R}^2$, let $p(K)$ denote the highest of the rightmost points of K, i.e., $p(K) = (a, b)$, where

$$a = \max\{x \mid (x, y) \in K \text{ for some } y\},$$

$$b = \max\{y \mid (a, y) \in K\}.$$

We claim that, almost surely, if A is nonempty, then $p(A)$ is a strictly n-multiple point of Z. This can be translated into the two following facts.

FACT$_1$. *Almost surely, if $A \neq \emptyset$, then $p(A) \notin Z(\mathbf{R}_+ \backslash I)$.*

FACT$_2$. *Almost surely, if $A \neq \emptyset$, then, for all $l \in \{1, \ldots, n\}$, $p(A)$ is simple ($\equiv$ strictly 1-multiple) for $Z_{|I_l}$* ($\equiv$ the restriction of Z to I_l).

Fact$_1$ is a consequence of

PROPOSITION$_1$. *Let X be a d-dimensional Brownian motion ($d \geqq 2$), let F be a closed subset of $\mathbf{R}_+$, and let v be a random variable measurable with respect to $\sigma(X_{|F})$, with values in $\mathbf{R}^d$, and such that if $0 \notin F$ then, almost surely, $v \neq X_0$. Then, almost surely, $v \notin X(\mathbf{R}_+ \backslash F)$.*

Observe that for an interval $]\alpha, \beta[$ included in $\mathbf{R}_+ \backslash F$, v is almost surely not in $X(]\alpha, \beta[)$.[†] But $\mathbf{R}_+ \backslash F$ is the union of the rational open subintervals of $\mathbf{R}_+$ that it contains.

To deduce Fact$_1$, we substitute Z, I, $p(A)$ for X, F, v, respectively. The fact that $p(A)$ is defined only if A is nonempty is of course of no importance. (We can, for instance, let $p(\emptyset)$ be any fixed point of $\mathbf{R}^2$.) But we have to show that $p(A)$ is measurable with respect to $\sigma(Z_{|I})$.

For a number $\delta > 0$, a subset of the plane is a δ-square if, and only if, it is of the form $[i\delta, (i+1)\delta] \times [j\delta, (j+1)\delta]$, where i and j are integers. Let A_δ denote the (finite) union of all δ-squares that intersect with A. It is a technical exercise to check that $p(A) = \lim_{\delta \to 0+} p(A_\delta)$ and that, for any δ-square R, the event $\{A$ encounters $R\}$ is in $\sigma(Z_{|I})$, so $p(A_\delta)$ is measurable with respect to $\sigma(Z_{|I})$. The measurability of $p(A)$ with respect to $\sigma(Z_{|I})$ follows immediately.

[†] A technical argument can go as follows. Note that v measurable with respect to $\Sigma \equiv \sigma(X_{|\mathbf{R}_+ \backslash]\alpha, \beta[})$. Let π be a version of $P(\cdot / \Sigma)$. For almost all ω, for any x in the plane, $\pi(x \in X(]\alpha, \beta[))(\omega) = 0$. Nothing forbids taking $x = v(\omega)$. All we have to do now is notice that the probability that v belongs to $X(]\alpha, \beta[)$ is the expectation of $\pi(v(\omega) \in X(]\alpha, \beta[))$.

Now we turn to Fact_2.

Let Y be a plane Brownian motion independent of Z. Fix some l in $\{1,\dots,n\}$, and let B denote the intersection of $Y(I_l)$ with the compact set $\bigcap_{m\in\{1,\dots,n\}\setminus\{l\}} Z(I_m)$, which is independent of Y. Fubini's theorem enables us to conclude that in order to establish Fact_2 it suffices to show that, almost surely, if B is nonempty, then $p(B)$ is a simple point of Y. So our problem is reduced to proving the following

PROPOSITION$_2$. *Let K be a compact subset of the plane. Then, almost surely, if $C \equiv Y([0,1]) \cap K$ is nonempty, Y admits $p(C)$ as a simple point.*

PROOF. For $t \in \mathbf{R}_+$, let

$$C_t = Y([0,t]) \cap K$$

and, on $\{C_t = \varnothing\}$, let

$$S_t = \inf\{s \geqq 0 \mid Y_s = p(C_t)\};$$

on $\{C_t = \varnothing\}$, let $S_t = 0$.

By the above definition, it is clear that, for all s in $[0, S_1[$, $Y_s \neq Y_{S_1}$.

Observe that, for $t \in \mathbf{R}_+$, Y_{S_t} is measurable with respect to $\sigma(Y_{|[0,t]})$, so, almost surely, $Y_{S_t} \notin Y(]t,\infty[)$. (Note that $(Y_{t+\cdot} - Y_t)_{\mathbf{R}+}$ is a plane Brownian motion independent of $(Y_\cdot)_{[0,t]}$, and the probability that it hits a given point over $]0,\infty[$ is zero.) So, almost surely, for all rational q in $\mathbf{R}_+$ (and, in particular, for rational q, such that $S_q = S_1$), $Y_{S_q} \notin Y(]q,\infty[)$. But $]S_1,\infty[= \bigcup_{q \text{ rational} > S_1}]q,\infty[$, and we deduce that, almost surely,

$$Y_{S_1} \notin Y([0, S_1[\cup]S_1, \infty[),$$

in which case Y_{S_1} is a simple point of Y.

All we have to do now is observe that if $Y([0,1])$ encounters K, then $p(C) = Y_{S_1}$.

COROLLARY. *Almost surely, the restriction of Z to any subset of $\mathbf{R}_+$ whose interior is nonempty has strictly n-multiple points, which are also strictly n-multiple points of Z.*

REMARKS. (1) Modifying very slightly the above considerations, one can prove that, almost surely, there exist points in the plane whose inverse image under Z is order-similar to N. Obviously, such points are strictly $\aleph_0$-multiple for Z. For strictly $\aleph_0$-multiple points whose inverse image is *bounded*, the technique of the present paper is probably insufficient. (See [1].)

(2) One is likely to feel that $(n+1)$-multiple points are "harder" to obtain than n-multiple ones, so "most" n-multiple points of Z are *not* $(n+1)$-multiple and *are*, consequently, strictly n-multiple.

A possible sense of the above "most" is a Hausdorff measure one. In a recent work, Le Gal ([3]) indicated a way of proving a conjecture of Taylor ([4]) to this effect. According to Taylor's conjecture, if M_n is the set of n-multiple points of Z, and if $h_\alpha(x) = x^2(\log 1/x)^\alpha$, then, almost surely, for all n, the h_α-Hausdorff measure of M_n is 0 if $\alpha \leqq n$, ∞ if $\alpha > n$.

Considering things from another angle, n-multiple points of Z may be expected to be "easier" to hit than $(n+1)$-multiple ones. Let $I_1, I_2, \ldots, I_n$ be mutually disjoint compact subsets of $\mathbf{R}_+$, and let X be a right-continuous random process defined on $\mathbf{R}_+$, taking values in the plane, independent of $Z_{|\cup I_k}$. Let $T = \inf\{t \geqq 0 \mid X_t \in \cap Z(I_k)\}$. Very slight modifications of our arguments show that, almost surely, if T is finite, then X_T is a strictly n-multiple point of Z.

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